Econ 101A: Notes

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Unconstrained Multivariable Optimization

$f(x_1, x_2..x_n) = f(\vec{x})$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$

0.1 First Order Conditions

$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ ... \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$

We need to have $\nabla f(\vec{x}) = 0$

This implies $\frac{\partial f}{\partial x_i} = 0 \; \forall i \in (1, n)$

0.2 Second Order Conditions

$\frac{\partial^2 f(x*)}{dx^2} < 0$

This implies a maximum, on the other hand we have $\frac{\partial^2 f(x*)}{dx^2} > 0$ is a minimum.

Hessian = H = $\begin{bmatrix} \frac{\partial^2 f}{\partial x_i \partial x_j} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_i^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & ... & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & .... & ... \\ ... & ... & ... & ... \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & ... & ... & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$ We have to evaluate this at every single $|H|_k$ which is the kxk sub matrix of H. One of three things could happen

1. If we have alternating signs with $|H|_1$ being negative, then $x*$ is a local maximum
2. If all sub determinants have positive signs, (¿ 0) then we have a local maximum
3. If some of the sub determinants are zeroes then additional analysis is required.

Implicit Function Theorem

Consider a function $x = g(p)$ and a point (p_0, x_0) that solves the implicit function
$f(p, x) = g(p) - x = 0$
If the following conditions holds:
f is continuously differentiable in a neighborhood of (p_0, x_0)
$f'(p_0, x_0) \neq 0$
Then the implicit function theorem tells us multiply useful things:

1. There is one and only one function $x = g(p)$ defined in the neighborhood of p_0 that satisfies $f(p, g(p)) = 0$ and $g(p_0) = x_0$
2. $\frac{\partial g(p)}{\partial x} = -\frac{f'_p(p,g(p))}{f'_x(p,g(p))}$

Constrained Optimization

$L(x_i, \lambda; p_i) = f(x_i) - \lambda \cdot h(x_i; p_i)$

For a constrained maximum, we essentially just apply the same techniques to the Lagrange and then go about solving from there.
There is nothing all to special about it, other than extra variables and a Hessian to account for this

Implicit Function Part 2

$f(\vec{p},\vec{x}) = 0$ This means that we have multiple functions of multiple variables, in which both f and $\vec{x}$ must have the same dimension
$\frac{\partial g_i}{\partial p_k} = -\frac{det\left[\frac{\partial(f_1..f_s)}{\partial(x_1..x_{i-1},p_k,x_{i+1},..x_s)}\right]}{det\left[\frac{\partial(f_1..f_s)}{\partial(x_1..x_{i-1},x_i,x_{i+1},..x_s)}\right]}$

Essentially the implicit function theorem says that if we have functions of multiple variables, and we want to find how one variable changes relative to another to another we can assume that there is a function $x = g(p)$ where both x and p are variables. Thus we can differentiate our original function by x and by p and use this to calculate first derivative. We don't need to explicitly know what the $g(p)$ just that it exists by some means

Envelope Theorem

$\frac{df(\vec{x}*,\vec{p})}{dp_k} = \frac{\partial f(\vec{x}^*,\vec{p})}{\partial x_1}\frac{x_1^*(p)}{\partial dp_k} + \frac{\partial f(\vec{x}^*,\vec{p})}{\partial x_2}\frac{x_2^*(p)}{\partial dp_k}..\frac{\partial f(\vec{x}^*,\vec{p})}{\partial x_n}\frac{x_n^*(p)}{\partial dp_k} + \frac{\partial f(\vec{x}^*,p)}{\partial dp_k}$ However, we know that by FOC that all but the last term equal zero thus we have simplification: $\frac{df(\vec{x}^*,\vec{p})}{dp_k} = \frac{\partial f(\vec{x}^*,p)}{\partial dp_k}$

Preference

0.3 Basics

1. She prefers x to $y \rightarrow x \succeq y$
2. She prefers y to $x \rightarrow y \succeq x$
3. Indifference between the two goods means, $x \sim y$

We also have difference between $\succeq$ and $\succ$ which means strongly prefer versus weakly prefer. $x \succ y \rightarrow x \succeq y$ and not $y \succeq x$
$x \sim y \rightarrow x \succeq y, y \succeq x$

0.4 Assumptions about Preferences

Axiom 1.1 Completeness: for all x and y either $x \succeq y$ or $y \succeq x$ or $x \sim y$. You can compare and rank all possibilities

Axiom 1.2 Transitivity: for all x, y and z if $x \succeq y$ and $y \succeq z$ then $x \succeq z$
Definition 1.1 Preferences are said to be **rational** if they satisfy completeness and transitivity
Definition 1.2 We say preferences are monotone if an individual always prefers bundles with more of **every** component
Definition 1.3 We say preferences are **strictly monotone** if an individual always prefers bundles with at least as much of every individual component and more of at least one component.
Definition 1.4: We say preferences are **convex** if for all x, y and $z \in X$ such that $x \succeq z$ and $y \succeq z$ then $\alpha x + (1-\alpha)y \succeq z$ for any $\alpha \in [0, 1]$

Utility

Definition 2.1 We say a function u: $X \to \mathbb{R}$ is a utility function representing a consumer's preferences if, for all $x, y \in X$

$$x \succeq y \iff u(x) \geq u(y) \tag{1}$$

Existence If $\succeq$ is rational and continuous on set $X \in \mathbb{R}^n$ there exists a continuous utility function u: $X \to \mathbb{R}$ that represents it
Utility is measured only relative to other utility quantities, because they aren't necessarily unique and possess no cardinal value

We can always transform our utility function by a strictly increasing function, because our utility is not unique and will preserve the relationship between them

Indifference Curves

Given a bundle, an **indifference curve** represents the set of bundles that a consumer is exactly indifferent between. Thus along any given indifference curve, the person gets the same amount of utility.
We graph several indifference curves and make an indifference map

0.5 Shape of Indifference Curves & the Marginal Rate of Substitution

marginal rate of substitution is defined as

$$MRS = \frac{\partial x_2}{\partial x_1} \tag{2}$$

On top of being downward slope we usually assume convexity, due to the fact that if we increase consumption of a good we should have less desire to consume that good

0.5.1 Calculating MRS

1. Direct Method: If you already have a equation representing $x_2(x_1, u_0)$ which is an explicit function of x_1, simply differentiate
2. IFT Method: Notice that by setting $u(x_1, x_2)$ equal to some u_0 you can use the implicit function theorem
   $\frac{\partial x_2}{\partial x_1} = -\frac{u'_{x_1}}{u'_{x_2}}$

Budget Constraints

$\sum_{i=1}^{N} p_i \cdot x_i \leq M$ where M is a consumer's income

Utility Maximization

0.6 Lagrangian Optimization

The main tool that we will use is Lagrangian optimization which allows us to to embed a consumer's budget constraint in their maximization problem. Suppose we have a utility function that we wish to maximize and a budget constraint $M = p_1x_1 + p_2x_2$

$$max L(x_1, x_2, \lambda) = u(x_1, x_2) - \lambda[p_1x_1 + p_2x_2 - M] \tag{3}$$

0.7 Equimarginal Principle (or the Tangency Condition)

It makes sense, as long as derive positive utility from items, that we maximize our spending (i.e spend our entire budget)
equimarginal principle at the optimum, the marginal utility of a dollar spent on good x_1 must equal the marginal utility of a dollar spent on good x_2

$$-\frac{p_1}{p_2} = -\frac{u'_{x_1}(x_1, x_2)}{u'_{x_2}(x_1, x_2)} = MRS \tag{4}$$

Essentially we want to choose the highest indifference curve while still abiding by the budget constraint. This only applies, however, to interior solutions, not to solutions that are lying on an axis (i.e zero of a particular good)

1 Indirect Utility

The indirect utility function takes price p and income M arguments and gives the maximum achievable utility. That is, shows the relationship between the value of the maximized utility function and the parameters of the problem

$$\begin{aligned} V(p, M) &= maxu(x, p)s.tp \cdot x \leq M \\ &= u(x(p, M)) \end{aligned}$$

2 Homogeneous Functions

A property of the indirect utility function that was mentioned in lecture is that it is "homogeneous of degree zero".
Definition 2.1 $f(\vec{x})$ is homogeneous of degree k if $f(\alpha\vec{x}) = \alpha^k f(\vec{x})$ for all nonzero α and all possible $\vec{x}$
Hence homogeneity of degree 0 means that
$f(\alpha\vec{x}) = f(\vec{x})$
Essentially multiplying all the variables in the utility function by the same constant will not affect the value

Comparative Statistics

Definition 3.1 The substitution effect is the change in the consumption of a good associated with a change in its price holding constant the level of utility
We can theoretically visualize this by moving the budget constraint to mirror the change in prices and then see the change in the bundle of goods.
Definition 3.2 The income effect is a change in the consumption of a good associated with a change real purchasing power, holding constant relative prices
We can see this change by increasing the budget constraint out meaning that we intersect a higher indifference curve

If the income effect is positive for a good then the good is referred to as a normal good
If the income effect is negative for a good then the good is referred to as an inferior good
If neither, it is a neutral good

If we happen to buy less of a good when prices go up, or if we buy more of a good when prices go down then this is referred to as a Giffen good. (very rarely plausible but still present)

Slutsky Equation

$$\frac{\partial x_i(\vec{p}, M)}{\partial dp_j} = \frac{\partial h_i(\vec{p}, u_0)}{\partial p_j} - \frac{\partial x_i(\vec{p}, M)}{\partial M} x_j(\vec{p}, M) \tag{5}$$

Expenditure Minimization and Comparative Statistics

We can look at minimizing the total expenditure to achieve a certain level of utility, which translates to:

$$minL(x_1, x_2, \lambda) = p_1x_1 + p_2x_2 - \lambda[u(x_1, x_2) - \bar{u}] \tag{6}$$

Complements and Substitutes

Gross Substitutes $\frac{\partial x_i}{\partial p_j} > 0$
Gross Complements $\frac{\partial x_i}{\partial p_j} < 0$
Net Substitutes $\frac{\partial h_i}{\partial p_j} > 0$
Net Complements $\frac{\partial h_i}{\partial p_j} < 0$

Labor Supply

Measures of Risk Aversion

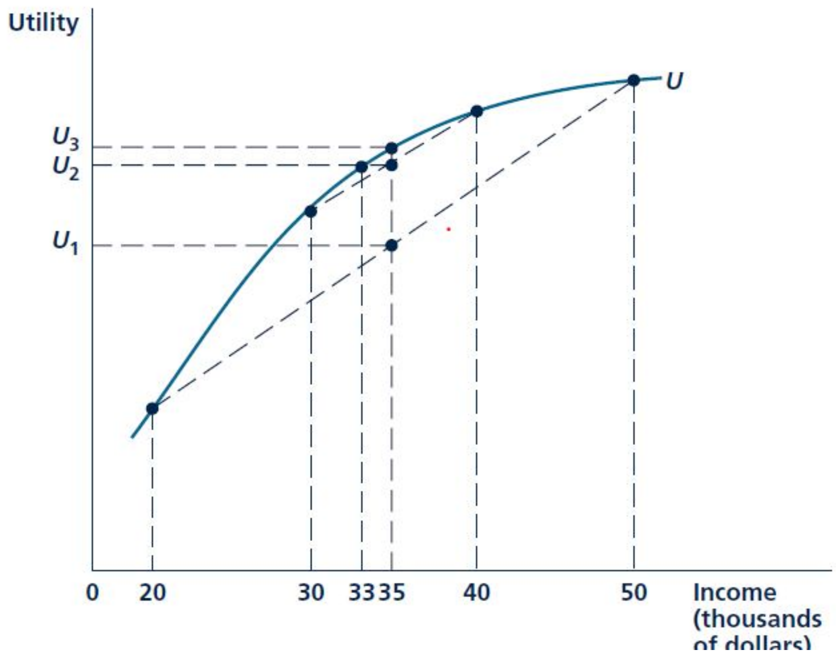
There are two measures of risk aversion:
Absolute risk aversion r_A:

$$r_A = -\frac{u''(x)}{u'(x)}$$

And relative risk aversion r_R

$$r_R = -\frac{u''(x)}{u'(x)}x$$

Individuals dislike uncertainty , and will try to mitigate that possibility as much as possible; the y would chose to have a steady income than a 50-50 chance that has the same expected income.
We can see that this holds with a concave $u(x)$ because if we connect the two possibilities of the gamble, plus and negative, then the average utility falls below the curve.



We can look at the example of Insurance:
We have wealth w and utility function $u(x)$ with $u'(x) > 0$ and $u''(x) < 0$.
There is a probability p of accident with loss L
It has the premium cost of q for each 1 paid in case of accident:
we get α units covered

$$max(1-p)u(w-q\alpha) + pu(w-q\alpha-L+\alpha)$$

The first term is the probability that we don't have the accident and the second term is the probability we do have the accident and the respective utilities that we get in each of the scenarios with a variable α
The first order condition:

$$\frac{u'(w-q\alpha)}{u'(w-q\alpha-L+\alpha)} = \frac{1-q}{q}\frac{p}{1-p}$$

This same time of principle can be applied to various other scenarios, specifically involving investment in a risky endeavor that has a certain probability of success, etc etc.

Time Consistency

Intertemporal choice:
Three periods: $t = 0$, $t = 1, t = 2$
At each period i, agents:

- having income $M_1' = M_i$ + savings/debt from the previous period
- choose consumption c_i
- Can save or borrow money $M_i' - c_i$
- cannot borrow in the remaining period

The utility function at $t = 0$:

$$u(c_0, c_1, c_2) = U(c_0) + \frac{1}{1+\delta}U(c_1) + \frac{1}{(1+\delta)^2}U(c_2)$$

Utility function at time $t = 1$ is:

$$u(c_1, c_2) = U(c_1) + \frac{1}{1+\delta}U(c_2)$$

Utility function at time $t = 2$ is;
$u(c_2) = U(c_2)$

We can derive the ratio for the preference of consumption in period 0 and 1 and the preference between period 1 and 2 and find that they are equal; entailing that the actions that will maximize during period 0 will also maximize in the future

Time Inconsistency

$u(c_t, c_{t+1}, c_{t+2}) = u(c_t) + \frac{\beta}{(1+\delta)}u(c_{t+1}) + \frac{\beta}{(1+\delta)^2}u(c_{t+2})...$
This causes a difference in the ratio between $t = 0, t = 1$ and $t = 1, t = 2$.
$\frac{U'(c_1*)}{U'(c_2*)} = \beta\frac{1+r}{1+\delta}$
$\frac{U'(c_1^{*,c})}{U'(c_2^{*,c})} = \frac{1+r}{1+\delta}$

These differ when we factor into the period 0 commitment, thus we consume too much early on and too little later one

Production Function

Production Functions describe the output firms can make given a set of inputs. We can put anything in input but usually restrict to L and K, labor and capital. All our production functions are in the long run. The difference, quantitatively, between short and run is that in short run there is at least one input that is fixed, while in long run all of them are flexible. Production Function: $y = f(\vec{z})$
Function: f: $\mathbb{R}^n \to R_+$
$\vec{z} = (z_1, z_2, z_3 ... z_n)$: labor, land, capital etc etc
Output y: Literally fucking anything, minivan
Properties of f:

- no free lunches: $f(0) = 0$
- positive marginal productivity: $f_i'(z) > 0$
- decreasing marginal productivity $f_i''(z) < 0$

2.1 Marginal Product and Average Product

We define the **marginal product of labor** as the change in output associated with employing an additional unit of labor, holding all else constant.

$$\frac{\partial f}{\partial L}$$

We define the **marginal product of capital** as the change in the output associated with employing an additional unit of capital, holding all else constant.

$$\frac{\partial f}{\partial K}$$

Isoquants essentially map the output for a given set of inputs on a graph (i.e if we have f dependent on L, K, we have a graph with L, K on axes and we draw the combinations of them needed for a certain fixed output f)

Isoquants $Q(y) = \vec{x}|f(\vec{x}) = y$
Set of inputs $\vec{z}$ required to produce y
Special case is when we have $z_1 = L$ labor and $z_2 = K$ which is capital
Isoquant: $f(L, K) - y = 0$
Slope of Isoquant $\frac{dK}{dL} = MRTS$ (marginal rate of technical substitution)
Expenditure on inputs: $wL + rK$
Firm objective function: $minwL + rK$ s.t $f(L, K) \geq y$

$$\frac{f'_L(L*, K*)}{f'_K(L*, K*)} = \frac{w}{r}$$

We can look at the derived demands for inputs:

- $L = L * (w, r, y)$
- $K = K * (w, r, y)$
- Value function at optimum is the cost function

$$c(w, r, y) = wL * (r, w, y) + rK * (r, w, y)$$

The firm's objective is to chose an optimal quantity of y given a price and the cost function

$$maxpy - c(w, r, y)$$

F.O.C
$p - c'(w, r, y) = 0$
This means that we have price equal to marginal cost to produce a unit
Second order is:
$-c''(w, r, y)$ which means that we have to have an increasing function otherwise we would have a minimum instead of maximum

Costs

The total cost is given by:

$$C = wL + rK$$

An **isocost** curve shows all possible combinations of labor and capital that can be purchased at a certain fixed cost. The slope of the isocost is $-\frac{w}{r}$ and the slope of the isoquant line is $-\frac{\partial K}{\partial L}$. The point at which we can produce the most for the lowest cost is when these two have the same slope, i.e the point of tangency.

Cost Minimization: Example

Lets assume we have the Cobb Douglas production curve: $y = f(L, K) = AK^{\alpha}L^{1-\alpha}$
We want to minimize:

$$wL + rK$$

where w is the cost of labor and r is the cost of renting a unit of capital
The return of scale depends on what $\alpha + \beta$ is equal to
The optimal amount of labor given that want to meet $f(L, K) = y$ is:

$$L*(w, r, y) = (\frac{y}{A})^{\frac{1}{\alpha+\beta}}(\frac{w}{r}\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}}$$

$$K*(w, r, y) = (\frac{y}{A})^{\frac{1}{\alpha+\beta}}(\frac{w}{r}\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}}$$

We can check a variety of comparative statistics:
$\frac{\partial L^*}{\partial A} < 0$ which means that as we increase technology we expect employment to decrease
$\frac{\partial L^*}{\partial y} > 0$ which means that more workers are needed to produce more output
$\frac{\partial L^*}{\partial w} < 0$ as wage goes up we expect that we would use less employees
etc etc
The total cost is therefore:

$$c(w, r, y) = wL^*(w, r, y) + rK^*(w, r, y)$$

$$c(w, r, y) = w(\frac{y}{A})^{\frac{1}{\alpha+\beta}}(\frac{w}{r}\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}} + r((\frac{y}{A})^{\frac{1}{\alpha+\beta}}(\frac{w}{r}\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}})$$

If w define

$$B = w(\frac{w}{r}\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}} + r(\frac{w}{r}\frac{\alpha}{\beta})^{-\frac{\beta}{\alpha+\beta}}$$

then we have our maximizing output as :

$$\max\ py - B(\frac{y}{A})^{\frac{1}{\alpha+\beta}}$$

Our first order condition is:

$$p - \frac{B}{A(\alpha+\beta)}(\frac{y}{A})^{\frac{1-\alpha-\beta}{\alpha+\beta}} = 0$$

The second order condition is :

$$-\frac{B(1-\alpha-\beta)}{A(\alpha+\beta)^2}(\frac{y}{A})^{\frac{1-2(\alpha+\beta)}{\alpha+\beta}}$$

If $\alpha + \beta = 1$, then we have:

- S.O.C is equal to zero
- That also means that our first order condition doesn't depend on y
- We want $p = \frac{B}{A}$ which means that if $p = \frac{B}{A}$ then we can choose any y
- If $p > \frac{B}{A}$ that means we should produce an infinite amount of y
- if $p < \frac{B}{A}$ that means that we should produce $y* = 0$

If $\alpha + \beta > 1$ then we have:

- S.O.C is positive
- That means that we become better off the more we produce thus we should produce an infinite amount

If $\alpha + \beta > 1$ then we have:

- S.O.C is negative
- That means we have an actual solution that can be solved.

2.2 Summary

: If a firm has the production function $f(K, L)$ the firm wants to find a $y = f(K, L)$ that would maximize the profit;

$$max_y R(y; p) - C(y; w, r)$$

This means that we want to maximize the revenue given a fixed p for revenue and fixed w, r for cost.
We can thus rewrite this as:

$$max_y y \cdot p - [r \cdot K^*(y; w, r) + w \cdot L^*(y; w, r)]$$

In order to solve this, we need to figure out the most optimal $K^*(y; w, r), L^*(y; w, r)$ such that for a given y they produce are the most efficient. This equates to solving;

$$min rK + wL \text{ s.t } f(K, L) = y$$

After we find the values of K^* and L^* we can then plug this into our original equation to maximize y and then solve from there.

3 Cost Curves

A cost function shows the lowest cost at which a firm can produce output y given a wage w and rental price r:

$$c(w, r, y) = [min_{L,K} rK + wL \text{ s.t } f(K, L) = y]$$

Marginal costs $MC = \frac{\partial c}{\partial y}$ → Cost Minimization

$$p = MC = \frac{\partial c(w, , r, y)}{\partial y}$$

The average cost is simply $\frac{c}{y}$
Thus in order to assess whether a firm breaks even, we just have to look at whether $py > c(w, r, y)$ which means that $p > AC$
Supply Function: Portion of marginal cost(the price equal marginal cost) above average cost

4 Supply Function

The supply function is $y^* = y^*(w, r, p)$
We know that the ideal y occurs at:
$p = c'(w, r, y)$ and thus $p - c'(w, r, y) = 0$
Thus our implicit function is:

$$\frac{\partial y^*}{\partial p} = -\frac{1}{-c''_{y,y}(w, r, y)}$$

We know that the cost function is convex and thus this has positive slope, thus as price increases the supple does too.

5 Perfectly Competitive Markets

Thus far we have focused on how firms choose inputs to minimize costs. Our goal is figure out how firms decide how much to produce and what to price it at. In a perfectly competitive market:

- *Product Homogeneity*: All firms in the market produce an identical product that is perfectly substitutable
- *Price Taking*: Firms are assumed to be small relative to the market and thus changing a single firm's price won't affect the market price
- *Free Entry and Exit* Firms can easily enter and exit (in the long run) and there is no barrier of entry
- *Full Information*: Consumers and firms know the structure of the market
- *Low Transaction Cost:* Consumers can easily buy from another firm

It is useful to analyze perfectly competitive markets because it will help us understand markets that are not perfectly competitive later on.

6 One-Step Profit Maximization

The goal is to maximize profit:
Perfect competition: p is given and set by the market.

- Firms are small relative to market
- Firms do not affect market price p_M

Revenue(R): $py = pf(L, K)$
Cost(C): $wL + rK$
Profit(P): R - C $= pf(L, K) - wL - rK$
Thus our goal is:

$$\max\ _{L,K} pf(L, K) - wL - rK$$

Thus our first order conditions are:
$\frac{\partial P}{\partial L} = pf'(L, K) - w = 0$
$\frac{\partial P}{\partial K} = pf'(L, K) - r = 0$
The second order condition is a Hessian that simplifies to:

$$p^2[f''_{L,L}f''_{K,K} - (f''_{L,K})^2] > 0$$

Thus as long as this is positive, usually reliant on $f''_{L,K}$ not being too large, we are fine.

7 Aggregation and Market Equilibrium

So far we have learned how to solve for a firm's supply functions and a consumer's demand function. Now we will discuss aggregation - we can add up the supply functions of different firms to obtain an aggregate market supply function and add up consumer's demand to get an aggregate market demand.

1. Solve each firm's supply function
2. Add up all the firms supply functions to get $Y_i^S(p_i, w, r)$
3. Solve for each consumer's demand function
4. Add up all the consumers demand functions to get $X_i^D(p_1..p_n, M_1...M_j)$

5. Set the demand function and the supply function equal to each other and then solve for the optimal value of p

6. Plug the optimal value of p into either of the equations and solve

If we have a single consumer then we assume that their utility is based on a variety of goods, x_1, x_2, where x_i represents the number of good of i they bought. Thus their utility is: $u(x_1, x_2..x_n)$ and they have prices $p_1, p_2....$ Thus to maximize utility we have to solve for optimal quantities which is dependent on:

$$x_1^* = x_1^*(p_1, ..p_n, M)$$
$$x_2^* = x_2^*(p_1, ..p_n, M)$$
$$....$$
$$x_n^* = x_n^*(p_1, ..p_n, M)$$

We can focus on good i and fix prices $p_1, p_2..p_{i-1}, p_{i+1}..p_n, M$ and thus we get the **Single-consumer demand function**:

$$x_i^* = x_i^*(p_i | p_1, p_2...p_{i-1}, p_{i+1}..M)$$

The sign of this depends on if good i is normal or positive. We would expect that if the price goes up it the quantity would go down if the good is normal, (negative slope) and could be negative or positive if the good is inferior.

If we have hella consumers (J) like a normal market then our market demand is equal to the total demand:

$$X_i(p_1, p_2..p_n, M^1, ..M^J) = \sum_{j=1}^{J} x_i^{j*}(p_1, p_2..p_n, M^j)$$

7.1 Market Equilibrium

The equilibrium price p_i is when there is no excess supply or excess demand.

$$Y^* = Y_i^S(p_i^*, w, r) = X_i^D(p_i^*, ...p_n^*, M^1..M^J)$$

7.2 Comparative Statistics of Equilibrium

Lets assume that we have an additional parameter α:

$$Y_i^S(p_i, w, r, \alpha)$$
$$X_i^D(\vec{p}, \vec{M}, \alpha)$$

We still have the same equilibrium but we can see how α affects it:

$$Y_i^S(p_i, w, r, \alpha) - X_i^D(\vec{p}, \vec{M}, \alpha = 0$$
$$\frac{dp^*}{d\alpha} = -\frac{\frac{\partial Y^S}{\partial \alpha} - \frac{\partial X^D}{\partial \alpha}}{\frac{\partial Y_i^S}{\partial p} - \frac{\partial X_i^D}{\partial p}}$$

Since we know that supply will always go up as price increase and demand will always go down, that means the denominator is positive and thus the sign of this depends on the numerator.

We can rewrite this using elasticities:

$$\frac{dp^*}{d\alpha}\frac{\alpha}{p} = -\frac{\frac{\partial Y^S}{\partial \alpha} - \frac{\partial X^D}{\partial \alpha}\frac{\alpha}{Y}}{\frac{\partial Y_i^S}{\partial p} - \frac{\partial X_i^D}{\partial p}\frac{p}{D}}$$

and use the fact that $Y = D$ in equilibrium to get:

$$\epsilon_{p,\alpha} = -\frac{\epsilon_{S,\alpha} - \epsilon_{D,\alpha}}{\epsilon_{S,p} - \epsilon_{D,p}}$$

8 Elasticities

These are normal measures of seeing how one variable changes in response to a change in another variable. The difference is that we represent it by percents, thus it becomes unit free. General Expression

$$\epsilon_{x,y} = \frac{\partial x}{\partial y}\frac{y}{x}$$

Price Elasticity of Individual Demand

$$\epsilon(x^*, p) = \frac{\partial x^*}{\partial p}\frac{p_x}{x}$$

Elasticity of Substitution:

$$\sigma = \frac{d(\frac{x_2}{x_1})}{d\frac{p_1}{p_2}} \cdot \frac{\frac{p_1}{p_2}}{\frac{x_2}{x_1}}$$

9 Taxation

Consider a tax t that is incorporated into price p. Since t is paid by the consumer but not received by the firm, the market equilibrium will no longer be at the point where the aggregate supply and demand functions intersect.

-New supply curve: $Y^S(p - t, w, r)$

-New demand curve: $X^D(p, M)$

This means that we have the intersection (our price equilibrium) at $Y^S - X^D = 0$. We can use differentiation to figure out that:

$$\frac{dp^*}{dt} = -\frac{-\frac{\partial Y^2}{\partial d}}{\frac{\partial Y^S}{\partial p} - \frac{\partial X^D}{\partial p}}$$

$$\frac{d(p^* - t)}{dt} = -\frac{-\frac{\partial X^D}{\partial d}}{\frac{\partial Y^S}{\partial p} - \frac{\partial X^D}{\partial p}}$$

We can use the elasticities to figure out whether the consumer or the producer bears more of the weight of the taxes; we can also calculate what the original price was, and what the new price for the consumer and the supplier is and thus see where the largest difference lies.

10 Surplus

10.1 Producer Surplus

Producer Surplus is just the profit:

$$\pi(p, y_0) = py_0 - c(y_0)$$

$$\pi(p, y_0) = y_0(p - \frac{c(y_0}{y_0}$$

which means that we have profit equal to the quantity minus the difference between profit and the average cost. OR using the identity $f(x) = f(0) + \int_0^x f'(x)dx$:

$$\pi = [p * 0 + p * \int_0^{y_0} 1dy] - [c(0) + \int_0^{y_0} c'(y)dy$$

$$\pi = \int_0^{y_0} (p - c'_y(y))dy - c_0$$

10.2 Consumer Surplus

We can measure consumer surplus as the total amount they were willing to spend versus the amount they actually spend. If we have $e(p, u)$ represent the expense given p prices and to achieve u utility, we can measure the consumer surplus by this difference.

11 Long Run Equilibrium

Firms will enter the market when there is profit to be made, which will cause the supply to increase, hence lowering the price. Since profits are directly proportionate to price, eventually the profit will reach zero and this is when people stop entering the market. We will only assume that firms can enter the market in the long run.

12 Monopoly: Price Maximization

In a monopoly we assume that there is a single firm that controls all supply and will maximize profits by setting p.

Setting prices implies either we choose p with the demand given by $y = D(p)$ or we set the quantity

The goal is to maximize profits:

$$max_y p(y)y - c(y)$$

$$p'(y)y + p(y) - c'(y) = 0$$

$$-p'(y)y = p(y) - c'(y)$$

$$-p'(y)\frac{y}{p} = \frac{p(y) - c'(y)}{p}$$

$$-\frac{1}{\epsilon_{y,p}} = \frac{p(y) - c'(y)}{p}$$

The R.H.S essentially is the price of good minus the cost of the good over the price, or the markup and the right hand is the inverse of the elasticity.

We have the quantity that the firm will produce when MR = MC (marginal revenue equals marginal cost) and thus we should see what the demand is at that point and charge accordingly. It is possible for the average cost to be higher than this price still and cause firm to run a deficit.

Generally the issue with monopolies remains that the prices are higher and the production tends to be lower. We can also consider the cases of price discrimination;

- **First Degree** This is when we can sell at different prices to differing consumers. Also known as perfect discrimination in which we charge people exactly what they are willing to pay and is ILLEGAL.

- **Second Degree** Price is a function of the quantity purchased, the same across all consumers. This is where we can bundle some quantities or sell different quantities, and is legal.

- **Third Degree** Segmented Markets: we charge consumers within each segment a certain price. Student discounts is an example, or selling certain products at higher price in certain countries.

13 Oligopoly

This is where there a few firms but not the level of perfect competition due to high barriers of entry

This is the most common type in the free market economy, such as soft drinks, car dealers, food manufacturers etc

Each firm maximizes:

$$max_{y_i} p(y_i + y_{-i})y_i - c(y_i)$$

where $y_{-i} = \sum_{j \neq i} y_j$

Because our profit depends on the prices of others who in turn depend on us we need Game Theory.

14 Game Theory

The study of strategic interaction between players. We have players $1...I$ and strategies for $s_i \in S_i$. The payoffs are $U(s_i, s_{-i})$

If we can find each player a dominant strategy, that it makes sense that all of them will take this dominant strategy. A dominant strategy is essentially a strategy that a player can take that will guarantee them the best payoff across every strategy FOR every other decision the other players make. This is represented formulaic by:

$$U_i(s_i^*, s_{-i}) \geq U_i(s_i, s_{-i}) \forall s_i \in S_i, \forall s_{-i} \in S_{-i}$$

There also exists Nash Equilibrium.

$$U_i(s_i^*, s_{-i}^*) \geq U_i(s_i, s_{-i}^*)$$

The Nash equilibrium is when each player can't achieve a better outcome assuming the other people keep their strategies the same. This only exists in mixed strategies, or where there is no player who's utility is entirely dependent on the disutility of another.

15 Oligopolies

15.1 Cournot

Let's assume that we have two firms cost $c_i(y_i) = cy_i, i = 1, 2$ Firms chose simultaneously quantity y_i and firm i maximizes:

$$max_{y_i} p(y_i + y_{-i})y_i - cy_i$$

which is just the revenue minus the expense, but the revenue is dependent on the price set by the market F.O.C:

$$p'_Y(y_i^* + y_{-1}^*)y_i^* + p - c = 0 \forall i$$

This is entirely dependent on nash equilibrium, which helps us solve for the total quantity. We can just solve for the optimal quantity of y_1 given y_2 and the optimal quantity of y_2 given y_1. Then we just solve the equations.

In this type of oligopoly, we price above the marginal cost.

15.2 Bertrand

In a Bertrand monopoly, firms choose the price and then produce quantity demanded by the market. If there are two firms then firm i has the following profit:

$$\pi_i(p_i, p_{-i}) = \begin{cases} (p_i - c)Y(p_i) & p_i < p_{-i} \\ (p_i - c)Y(p_i)/2 & p_i < p_{-i} \\ 0 & p_i > p_{-i} \end{cases}$$

$p_1 = p_2 = c$ is a Nash equilbrium, because if we assume that firm i holds its price constant, then if the other firm increases its price, it gets zero, and if the other firm decreases its price it runs negative profit.

To show that something is in equilbrium, we have to show that there is NO profitable deviation and to show that something is not an equilibrium we can show that there is any profitable deviation.

The Bernard oligopoly shows that we have to have pricing that is equivalent to the perfect competition, pricing at marginal cost.